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Time-evolution properties of a linear Boltzmann collision operator

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Abstract. Some properties of a linear Boltzmann collision operator acting in the L^1 space of absolutely integrable functions of the velocity are derived. The system considered consists of particles moving in a dilute equilibrium gas. The case of a constant accelerating force acting upon the particles (as encountered in electron swarm experiments) is also studied. It is found that the collision operator is a dissipative operator which generates a strongly continuous contraction semi-group. It is also shown that the time evolution leaves the positivity and normalisation of the distribution function invariant.

1. Introduction

In the present work we study the properties of the linearised collision operator J for a binary mixture of dilute gases made up of point particles. It is assumed that the concentration of the first component (1) is so low that the second component (2) remains in equilibrium. Typical examples of this situation are encountered in electron and ion swarm experiments.

The time dependence of the distribution function $f_1(\boldsymbol{x}, \boldsymbol{v}_1, t)$ of the first component is supposed to be described by the linearised Boltzmann equation for a spatially infinite system;

$$(\partial_t + \boldsymbol{v}_1 \cdot \partial_{\boldsymbol{x}} + \boldsymbol{a} \cdot \partial_{\boldsymbol{v}_1}) f_1(\boldsymbol{x}, \boldsymbol{v}_1, t) = (Jf_1)(\boldsymbol{x}, \boldsymbol{v}_1, t). \tag{1.1}$$

Here $\boldsymbol{x}$ and t are the space and time variables, $\boldsymbol{v}_1$ is the velocity variable for component 1 and J is the linearised collision operator. In swarm experiments a constant homogeneous electric field is applied to the system. This gives rise to a constant acceleration $\boldsymbol{a}$.

The interest in (1.1) lies in its use for the description of irreversible processes in dilute gases. Since the irreversible behaviour enters through the collision operator J in (1.1) it is important to study its properties such as the location of its spectrum and its dissipative nature.

The usual procedure is to put the problem in a Hilbert space setting by means of the following procedure: f_1 is written as

$$f_1(\boldsymbol{x}, \boldsymbol{v}_1, t) = f_1^{(0)}(\boldsymbol{v}_1)\phi(\boldsymbol{x}, \boldsymbol{v}_1, t), \tag{1.2}$$

where $f_1^{(0)}$ is the Maxwell–Boltzmann distribution function taken at the temperature T

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of the second component (k is Boltzmann's constant and m_1 the mass of a particle of component 1).

$$f_1^{(0)}(\boldsymbol{v}_1)=[m_1/(2\pi kT)]^{3/2}\exp[-m_1v_1^2/(2kT)]. \tag{1.3}$$

The inner product

$$(\Phi,\Psi)=\int \mathrm{d}\boldsymbol{v}_1\, f_1^{(0)}(\boldsymbol{v}_1)\Phi(\boldsymbol{v}_1)\bar{\Psi}(f_1) \tag{1.4}$$

defines a Hilbert space $\mathscr{H}$. The collision operator C acting in $\mathscr{H}$ is now defined by the relation

$$Jf=Jf_1^{(0)}\Phi=f_1^{(0)}C\Phi. \tag{1.5}$$

C is a non-positive, densely defined, symmetric operator. It follows that C has a self-adjoint extension $\bar{C}$ acting in $\mathscr{H}$. Now the well developed apparatus of Hilbert space operator theory can be applied in order to study the properties of C. This has been done in the past for related problems where the interaction between the particles was taken to be a repulsive power potential (Pao 1974). Within the context of neutron thermalisation problems, hard sphere interactions have also been considered (Kuščer and Corngold 1965). In the first case an additional complication arises since the classical total cross section is not finite for repulsive power potentials.

The above approach leaves something to be desired. The reason is that the topology of the Hilbert space $\mathscr{H}$ is not the natural one for the present situation (see also the remarks made by Uhlenbeck and Ford (1963, p 88)). Clearly the choice of a topology for the underlying function space should be based upon physical considerations. Let us therefore consider the number density $n_1(\boldsymbol{x},t)$ of component 1 as given by

$$n_1(\boldsymbol{x},t)=\int \mathrm{d}\boldsymbol{v}_1\, f_1(\boldsymbol{x},\boldsymbol{v},t). \tag{1.6}$$

Let V be a volume in coordinate space with finite Lebesgue measure $\mu(V)$. The average number of particles of type 1 to be found in V at time t is

$$N_V(t)=\int_V \mathrm{d}\boldsymbol{x}\, n_1(\boldsymbol{x},t). \tag{1.7}$$

Since N_V must be finite for such V it follows that f should be Lebesgue-integrable over $\boldsymbol{v}$ and locally Lebesgue-integrable over $\boldsymbol{x}$. Thus, in order to give (1.1) a meaning it is necessary to introduce a topology consistent with the above integrability conditions. The integrability over $\boldsymbol{v}$ causes no problems; we have to require that f as a function of $\boldsymbol{v}$ is absolutely integrable, i.e. an element of $L^1(\mathbb{R}^3,\mathrm{d}\boldsymbol{v})$ in the absence of any $\boldsymbol{x}$ dependence. This is the case in spatially homogeneous systems. Since J only acts on the velocity variable, a system will remain spatially homogeneous once this is the case at some initial time. This situation is not altered by the presence of an acceleration term $\boldsymbol{a}\cdot\partial\boldsymbol{v}_1$ with $\boldsymbol{a}$ independent of $\boldsymbol{x}$.

Another simplifying situation is that of a finite number of particles of type 1. Then the local integrability over $\boldsymbol{x}$ can be replaced by global integrability. $f_1(\boldsymbol{x},\boldsymbol{v}_1,t)\in L^1(\mathbb{R}^6,\mathrm{d}\boldsymbol{v}_1\,\mathrm{d}\boldsymbol{x})$. In fact the experimental situation encountered in swarm experiments can be of this type. At some initial time a finite number of electrons is introduced (for instance by photoionisation caused by a pulse of photons produced by a laser). As long as no secondary electrons are produced by collisions with the atoms or molecules of component 2 or electrons are captured by these particles, the number of electrons will remain constant. Of course no stationary situation can build up in this way; the electron density will eventually decay to zero.

In the general case $f_1(\boldsymbol{x}, \boldsymbol{v}_1)$ is, upon integration over $\boldsymbol{v}_1$, mapped into the space $\mathscr{F}$ of locally integrable functions of $\boldsymbol{x}$, i.e. $f_1 \in L^1(\mathscr{F}, \mathbb{R}^3, \mathrm{d}\boldsymbol{v}_1)$. $\mathscr{F}$ can be topologised by means of a suitable set of semi-norms so that it becomes a Fréchet space. This procecure is entirely compatible with the physical situation at hand, but unfortunately the theory of spaces of the type $L^1(\mathscr{F}, \mathbb{R}^3, \mathrm{d}\boldsymbol{v})$ is not well developed.

The situation is different if $\mathscr{F}$ is a Banach space (Dunford and Schwartz 1957, chap IV). Of course, other approaches are possible, for instance by imposing suitable initial or boundary conditions. Then, however, we are no longer dealing with the general infinite space problem.

It will be clear that, whatever the situation is, it makes sense to study the properties of J or, in the presence of an accelerating force, $J(\boldsymbol{a}) = J - \boldsymbol{a} \cdot \partial \boldsymbol{v}_1$ as an operator in the Banach space $\mathscr{X} = L^1(\mathbb{R}^3, \mathrm{d}\boldsymbol{v}_1)$. This corresponds to the study of C in $\mathscr{H}$, but as discussed in the foregoing the choice of topology made here seems to have a better physical motivation. Also, as mentioned in § 3, serious problems occur with this approach if a non-vanishing acceleration term is present in (1.1).

Throughout this work we assume that the differential cross section $\sigma(\boldsymbol{g}, \boldsymbol{g}')$ (see § 2) entering into the collision operator J is an essentially bounded measurable function of $\boldsymbol{g}$ and $\boldsymbol{g}'$. As a consequence the total cross section is finite almost everywhere. We also make the usual assumption that $\sigma(\boldsymbol{g}, \boldsymbol{g}')$ possesses the symmetry property $\sigma(\boldsymbol{g}, \boldsymbol{g}') = \sigma(\boldsymbol{g}', \boldsymbol{g})$.

In § 2 we study the existence problem of the collision operator J and its adjoint J^*, whereas in § 3 we show that these operators (and also $J(\boldsymbol{a}) = J - \boldsymbol{a} \cdot \partial_v$ and its adjoint) are dissipative operators which generate strongly continuous contraction semi-groups. There it is also found that the positivity and normalisation of the distribution function $f_1(\boldsymbol{v}_1)$ are preserved in time.

In order to obtain these results use has been made of the notion of a semi-inner product (Lumer 1961). Thus it becomes possible to define the numerical range of an operator in a general Banach space and to define a dissipative operator in terms of the properties of its numerical range. Further results by Lumer and Phillips (1961) then connect dissipative operators with generators of strongly continuous contraction semi-groups.

For the various mathematical concepts used in these sections we refer to Dunford and Schwartz (1957), Hille and Phillips (1957), Kato (1966) and Yosida (1966). The relevant physical literature concerning the Boltzmann equation can be found in Chapman and Cowling (1970) and Waldmann (1958).

2. The existence of J and J^*

Let $\mathscr{X} = L^1(\mathbb{R}^3, \mathrm{d}\boldsymbol{v}_1)$ be the space of complex integrable functions of $\boldsymbol{v}_1 \in \mathbb{R}^3$. Its dual space is $\mathscr{X}^* = L^\infty(\mathbb{R}^3, \mathrm{d}\boldsymbol{v}_1)$, the space of Lebesgue measurable essentially bounded functions. Let $f \in \mathscr{X}$ and $\phi \in \mathscr{X}^*$. Their respective norms are

$$\|f\|_1 = \int \mathrm{d}\boldsymbol{v}_1 |f(\boldsymbol{v}_1)|, \qquad \|\phi\|_\infty = \operatorname{ess\,sup} |\phi(\boldsymbol{v}_1)|. \tag{2.1}$$

A bounded linear functional $\Phi(f)$, $f \in \mathscr{X}$ can be represented as

$$\Phi(f) = \int \mathrm{d}\boldsymbol{v}_1\, f(\boldsymbol{v}_1)\bar{\phi}(\boldsymbol{v}_1) \equiv \langle f, \phi \rangle \tag{2.2}$$

where ϕ is an element of $\mathscr{X}^*$. Note that $\langle f, \phi\rangle$ is linear in its first entry and conjugate linear in its second. The Hilbert space $\mathscr{H}$ introduced in § 1 is $\mathscr{H} = L^2(\mathbb{R}^3, f_1^{(0)}(\boldsymbol{v}_1)\,\mathrm{d}\boldsymbol{v}_1)$. $f_1^{(0)}(\boldsymbol{v}_1)$ as given by (1.3) is a positive element of $L^2(\mathbb{R}^3, \mathrm{d}\boldsymbol{v}_1)$ and

$$\|f_1^{(0)}\|_1 = \int \mathrm{d}\boldsymbol{v}_1\, f_1^{(0)}(\boldsymbol{v}_1) = 1. \tag{2.3}$$

We denote the norm on $\mathscr{H}$ by $\|\phi\|_2 \equiv (\phi, \phi)^{1/2}$, $\phi \in \mathscr{H}$.

Let $\phi \in \mathscr{X}^*$. Then

$$\|\phi\|_2^2 = \int \mathrm{d}\boldsymbol{v}_1\, f_1^{(0)}(\boldsymbol{v}_1)|\phi(\boldsymbol{v}_1)|^2 \leq \|\phi\|_\infty^2 \|f_1^{(0)}\|_1 = \|\phi\|_\infty^2$$

so that $\phi \in \mathscr{H}$. Now let $\phi \in \mathscr{H}$. Then

$$\|f^{(0)}\phi\|_1 = \int \mathrm{d}\boldsymbol{v}_1\, f_1^{(0)}(\boldsymbol{v}_1)|\phi(\boldsymbol{v}_1)| \leq \left(\int \mathrm{d}\boldsymbol{v}_1\, f_1^{(0)}(\boldsymbol{v}_1)|\phi(\boldsymbol{v}_1)|^2\right)^{1/2} \cdot \left(\int \mathrm{d}\boldsymbol{v}_1\, f_1^{(0)}(\boldsymbol{v}_1)\right)^{1/2} = \|\phi\|_2,$$

i.e. for $\phi \in \mathscr{H}$, $f^{(0)}\phi \in \mathscr{X}$. We summarise these results in the following proposition.

Proposition 2.1. $\mathscr{X}^* \subset \mathscr{H}$ i.e. $\phi \in \mathscr{X}^* \Rightarrow \phi \in \mathscr{H}$. Also $\phi \in \mathscr{H} \Rightarrow f^{(0)}\phi \in \mathscr{X}$ and $\|f^{(0)}\phi\|_1 \leq \|\phi\|_2 \leq \|\phi\|_\infty$.

The explicit form of the collision operator J is given by

$$(Jf_1)(\boldsymbol{v}_1) = \int \mathrm{d}\boldsymbol{v}_2 \int \mathrm{d}\Omega_{\boldsymbol{g}'}\, g\sigma(\boldsymbol{g}; \boldsymbol{g}') \cdot [f_2^{(0)}(\boldsymbol{v}_2')f_1(\boldsymbol{v}_1') - f_2^{(0)}(\boldsymbol{v}_2)f_1(\boldsymbol{v}_1)]. \tag{2.4}$$

Here $f_2^{(0)}(\boldsymbol{v}_2)$ is the Maxwell–Boltzmann distribution function at the temperature T of the second component

$$f_2^{(0)}(\boldsymbol{v}_2) = n_2[m_2/(2\pi kT)]^{3/2} \exp[-m_2 v_2^2/(2kT)]. \tag{2.5}$$

Here k is Boltzmann's constant, n_2 the uniform density of component 2 and m_j is the mass of a particle of component j. In (2.4) primed variables have their usual meaning as post-collisional velocities. $\boldsymbol{g} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $\boldsymbol{g}' = \boldsymbol{v}_1' - \boldsymbol{v}_2'$ and $\sigma(\boldsymbol{g}; \boldsymbol{g}')$ is the differential cross section for mutual scattering of particles of type 1 and type 2. Momentum and energy conservation lead to the relations

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = m_1\boldsymbol{v}_1' + m_2\boldsymbol{v}_2', \qquad g = g'. \tag{2.6}$$

The integral $\int \mathrm{d}\Omega_{\boldsymbol{g}'}$ is over the angles of $\boldsymbol{g}'$. We can write (2.4) as

$$(Jf_1)(\boldsymbol{v}_1) = \int \mathrm{d}\boldsymbol{v}_1'\, K(\boldsymbol{v}_1, \boldsymbol{v}_1')f_1(\boldsymbol{v}_1') - \nu(\boldsymbol{v}_1)f_1(\boldsymbol{v}_1). \tag{2.7}$$

Here

$$\nu(\boldsymbol{v}_1) = \int \mathrm{d}\boldsymbol{v}_2\, |\boldsymbol{v}_1 - \boldsymbol{v}_2|\sigma_{\text{tot}}(|\boldsymbol{v}_1 - \boldsymbol{v}_2|)f_2^{(0)}(\boldsymbol{v}_2), \tag{2.8}$$

where the total cross section is given by

$$\sigma_{tot}(g)=\int d\Omega_{g'}\sigma(\boldsymbol{g};\boldsymbol{g}'). \tag{2.9}$$

Furthermore (see Waldmann 1958, § 30)

$$K(\boldsymbol{v}_1,\boldsymbol{v}_1')=\int d\boldsymbol{v}_2\,\sigma(\boldsymbol{g};\boldsymbol{g}')\delta(\tfrac{1}{2}g^2-\tfrac{1}{2}g'^2)f_2^{(0)}(\boldsymbol{v}_2'). \tag{2.10}$$

As shown in the Appendix, we can write

$$K(\boldsymbol{v}_1,\boldsymbol{v}_1')=n_2\bar{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}_1')(2\Delta)^{-1}(\alpha/\pi)^{1/2}\exp[-\alpha\Delta^{-2}(\Delta^2+\boldsymbol{\Delta}\cdot\boldsymbol{v}_1')^2] \tag{2.11}$$

where $\alpha=m_2/(2kT)$, $\boldsymbol{\Delta}=[(m_1+m_2)/(2m_2)]\,(\boldsymbol{v}_1-\boldsymbol{v}_1')$ and

$$\bar{\sigma}(\boldsymbol{v}_1,\boldsymbol{v}_1')=(\alpha/\pi)\int du_1\,du_2\exp[-\alpha(u_1^2+u_2^2)].$$

$$\sigma(u_1+v_{11},u_2+v_{12},\Delta;u_1+v_{11},u_2+v_{12},-\Delta). \tag{2.12}$$

Here v_{ij} is the jth component of $\boldsymbol{v}_i$. For a classical hard sphere interaction $\sigma(\boldsymbol{v}_1,\boldsymbol{v}_1')=\sigma_{hs}$ is independent of the velocities, in which case

$$\bar{\sigma}_{hs}=\sigma_{hs},$$

$$K_{hs}(\boldsymbol{v}_1,\boldsymbol{v}_1')=n_2\sigma_{hs}(2\Delta)^{-1}(\alpha/\pi)^{1/2}\exp[-\alpha\Delta^{-2}(\Delta^2+\boldsymbol{\Delta}\cdot\boldsymbol{v}_1')^2] \tag{2.13}$$

and

$$\nu_{hs}(\boldsymbol{v}_1)=2n_2\sigma_{hs}(\alpha/\pi)^{1/2}\biggl(v_1\int_0^{v_1}dw\exp(-\alpha w^2)$$

$$+v_1^{-1}\int_0^{v_1}dw\,w^2\exp(-\alpha w^2)+\alpha^{-1}\exp(-\alpha v_1^2)\biggr). \tag{2.14}$$

Thus

$$\nu_{hs}(\boldsymbol{v}_1)\sim 2n_2\sigma_{hs}v_1,\qquad v_1\to\infty. \tag{2.15}$$

An easy computation shows that $\partial_{v_1}\nu_{hs}(v_1)\geq 0$ so that $\nu_{hs}(v_1)$ is an increasing function of v_1 with minimum

$$\nu(0)=2n_2\sigma_{hs}(\pi\alpha)^{-1/2}. \tag{2.16}$$

As mentioned in the Introduction, we confine ourselves to systems with finite differential and total cross sections. Thus ν and K, the operators associated with the multiplicative function $\nu(\boldsymbol{v}_1)$ and the kernel $K(\boldsymbol{v}_1,\boldsymbol{v}_1')$ respectively, have a meaning as separate objects (contrary to the classical repulsive power case where the total cross section does not exist). Nevertheless, as brought out by the example of hard spheres, these operators need not be bounded. (The operator $\nu_{hs}(v_1)$ is clearly unbounded.) This raises the question whether (2.4) and (2.7) make any sense at all. In order to proceed, it is convenient to consider the action of ν and K on simple functions. In our case these functions are the finitely valued (values $\alpha_1,\ldots,\alpha_n$) functions $f(\boldsymbol{v})$ from $\mathbb{R}^3$ to $\mathbb{C}$ with $B_i=\{\boldsymbol{v}|f(\boldsymbol{v})=\alpha_i\}$ Lebesgue measurable. Thus, if χ_B denotes the characteristic function for the set B the simple functions can be written as

$$f(\boldsymbol{v})=\sum_{i=1}^{n}\alpha_i\chi_{B_i}(\boldsymbol{v}). \tag{2.17}$$

Without loss of generality we may assume that B_i has non-zero Lebesgue measure. The simple functions form a dense set in $L^p(\mathbb{R}^3, d\boldsymbol{v})$, $1 \leq p \leq \infty$. (For $1 \leq p < \infty$ see Dunford and Schwartz (1957, p 105) and for $p = \infty$ see Yosida (1966, p 118).) The subset of simple functions with bounded support (so that $f(\boldsymbol{v}) = 0$ for $v \equiv |\boldsymbol{v}| > \rho$ for some finite positive ρ) will be referred to as the set of b-simple functions. The b-simple functions are dense in $L^p(\mathbb{R}^3, d\boldsymbol{v})$, $1 \leq p < \infty$ but not in $L^\infty(\mathbb{R}^3, d\boldsymbol{v})$.

Proposition 2.2. Let $\mathcal{M} \subset \mathcal{X}^*$ be the linear subset $\mathcal{M} = \{\phi \in \mathcal{X}^* = L^\infty(\mathbb{R}^3, d\boldsymbol{v}) | \lim_{\rho\to\infty} \operatorname{ess\,sup}_{v>\rho} |\phi(\boldsymbol{v})| = 0\}$. Then

(a) $\mathcal{M}$ is a closed subspace of $\mathcal{X}^*$.

(b) The b-simple functions are dense in $\mathcal{M}$.

(c) $\mathcal{X}$ can be associated with a closed linear subspace $\mathcal{K}$ of $\mathcal{M}^*$ by defining ($\phi \in \mathcal{M}$, $f \in \mathcal{X}$) $\Phi_f(\phi) = \langle \phi, f \rangle$. In addition $\|\Phi_f\| \equiv \sup\{|\langle \phi, f\rangle|, \phi \in \mathcal{M}, \|\phi\|_\infty = 1\} = \|f\|_1$.

Proof.

(a) Let $\{\phi_n\} \subset \mathcal{M}$ be a Cauchy sequence with respect to the norm-topology in $\mathcal{X}^*$. Then $\phi_n \to \phi \in \mathcal{X}^*$. For almost every $\boldsymbol{v}$ we have $|\phi(\boldsymbol{v})| \leq |\phi(\boldsymbol{v}) - \phi_n(\boldsymbol{v})| + |\phi_n(\boldsymbol{v})| \leq \|\phi - \phi_n\|_\infty + |\phi_n(\boldsymbol{v})|$. For given $\varepsilon > 0$ there is an $n_0 = n_0(\varepsilon)$ such that the first term on the right is smaller than ε for $n > n_0$. Keeping $n > n_0$ fixed from now on, we can now make the second term on the right smaller than ε (for almost every $\boldsymbol{v}$) by taking $\boldsymbol{v}$ sufficiently large. Thus $\phi \in \mathcal{M}$.

(b) For given $\phi(\boldsymbol{v}) \in \mathcal{M}$ let $\phi_\rho(\boldsymbol{v}) = \phi(\boldsymbol{v})$, $v \leq \rho$ and $\phi_\rho(\boldsymbol{v}) = 0$ for $v > \rho$. Then $\|\phi\|_\infty \leq \|\phi_\rho\|_\infty + \|\phi - \phi_\rho\|_\infty$. The last term on the right can be made arbitrarily small by taking ρ sufficiently large. Keeping ρ fixed from now on, we can find a b-simple function ψ which vanishes for $v > \rho$ and for which $\|\phi_\rho - \psi\|$ is arbitrarily small. Thus $\|\phi - \psi\|_\infty \leq \|\phi - \phi_\rho\|_\infty + \|\phi_\rho - \psi\|_\infty$ and the right-hand side can be made arbitrarily small.

(c) Clearly $|\Phi_f(\phi)| = |\langle \phi, f\rangle| \leq \|\phi\|_\infty \|f\|_1$ so that $\|\Phi_f\| \leq \|f\|_1$. Let $\phi_f(\boldsymbol{v}) = f(\boldsymbol{v})/|f(\boldsymbol{v})|$ for $\boldsymbol{v}$ with $f(\boldsymbol{v}) \neq 0$ and let $\phi_f(\boldsymbol{v}) = 0$ otherwise. Further, let $G_n(\boldsymbol{v}) = 1$ for $v \leq n$ and $G_n(\boldsymbol{v}) = 0$ for $v > n$, $n = 1, 2, 3 \ldots$. Then $\phi_n(\boldsymbol{v}) = G_n(\boldsymbol{v})\phi_f(\boldsymbol{v})$ is contained in $\mathcal{M}$ and $\|\phi_n\|_\infty = 1$. Now

$$\|\Phi_f\| \geq \sup_n |\Phi_f(\phi_n)| = \sup_n \left| \int d\boldsymbol{v}\, G_n(\boldsymbol{v})\phi_f(\boldsymbol{v})\bar{f}(\boldsymbol{v}) \right| = \sup_n \int_{v\leq n} d\boldsymbol{v}\, |f(\boldsymbol{v})| = \|f\|_1,$$

so that we must have $\|\Phi_f\| = \|f\|_1$. From this result it is clear that $\mathcal{K}$ is closed. In the following we shall no longer distinguish between $\mathcal{K}$ and $\mathcal{X}$ if no confusion can arise.

We now return to the operators ν and K. Let $f(\boldsymbol{v}_1)$ be b-simple. Thus f vanishes for $v > \rho$ for some $\rho > 0$. Let $\|\sigma_{\text{tot}}\|_\infty = \{\operatorname{ess\,sup} \sigma_{\text{tot}}(g), g \geq 0\}$. Then

$$\nu(\boldsymbol{v}_1) \leq \|\sigma_{\text{tot}}\|_\infty \int d\boldsymbol{v}_2 |\boldsymbol{v}_1 - \boldsymbol{v}_2| f_2^{(0)}(\boldsymbol{v}_2) \tag{2.18}$$

and

$$\|\nu f\|_1 \leq \|\sigma_{\text{tot}}\|_\infty \left\{ \sup_{v_1\leq\rho} \int d\boldsymbol{v}_2\, |\boldsymbol{v}_1 - \boldsymbol{v}_2| f_2^{(0)}(\boldsymbol{v}_2) \right\} \|f\|_1. \tag{2.19}$$

Since the right-hand side of (2.19) is finite it follows that the b-simple functions are in the domain of ν, i.e. ν is densely defined. It is straightforward to show, by making estimates of the type (2.18), that $\nu(\boldsymbol{v})$ is continuous. The maximal domain of ν is $\mathcal{D}(\nu) = \{f \in \mathcal{X} | \nu f \in \mathcal{X}\}$. Since $\nu(\boldsymbol{v}) \geq 0$ the operator $(1+\nu)^{-1}$ is a bounded multiplication operator in $\mathscr{X}$. Consequently $1+\nu$ and hence ν with domain $\mathscr{D}(\nu)$ is closed. K possesses the property (see the Appendix)

$$\int \mathrm{d}\boldsymbol{v}\, K(\boldsymbol{v}, \boldsymbol{v}') = \nu(\boldsymbol{v}'). \tag{2.20}$$

Thus, for $f(\boldsymbol{v}) \in \mathscr{D}(\nu)$,

$$\begin{aligned} \|Kf\|_1 &= \int \mathrm{d}\boldsymbol{v} \left| \int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}, \boldsymbol{v}') f(\boldsymbol{v}') \right| \\ &\leqslant \int \mathrm{d}\boldsymbol{v} \int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}, \boldsymbol{v}') |f(\boldsymbol{v}')| \\ &= \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\boldsymbol{v}\, K(\boldsymbol{v}, \boldsymbol{v}') |f(\boldsymbol{v}')| = \|\nu f\|_1. \end{aligned} \tag{2.21}$$

Here the Fubini–Fonelli theorem was used in order to interchange the integrations. Since $K(\boldsymbol{v}, \boldsymbol{v}')$ is non-negative, equality in (2.21) holds for non-negative $f(\boldsymbol{v})$. Thus K is relatively ν-bounded with relative bound one. We conclude that

$$J = K - \nu, \qquad \mathscr{D}(J) = \mathscr{D}(\nu), \tag{2.22}$$

is well defined. If ν is bounded it follows from (2.21) that K is bounded as well and hence J is bounded. In general, however, J may not even be closed.

It follows from (2.20) that for $f \in \mathscr{D}(J)$

$$\mathrm{d}\boldsymbol{v}(Jf)(\boldsymbol{v}) = \langle Jf, 1 \rangle = 0. \tag{2.23}$$

The physics behind this is that no particles of component 1 are created or annihilated during collisions. Since J is densely defined it follows that it has a uniquely defined closed adjoint J^* acting in $\mathscr{X}^*$. For $f \in \mathscr{D}(J)$ and $\phi \in \mathscr{D}(J^*)$ we have

$$\langle Jf, \phi \rangle = \langle f, J^* \phi \rangle. \tag{2.24}$$

Now let $f \in \mathscr{D}(J)$ and let ϕ be b-simple. Using the Fubini–Tonelli theorem and relabelling the variables $(\boldsymbol{v} \leftrightarrow \boldsymbol{v}')$, we obtain

$$\begin{aligned} \langle Jf, \phi \rangle &= \int \mathrm{d}\boldsymbol{v} \Big(\int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}, \boldsymbol{v}') f(\boldsymbol{v}') \bar{\phi}(\boldsymbol{v}) - \nu(\boldsymbol{v}) f(\boldsymbol{v}) \Big) \bar{\phi}(\boldsymbol{v}) \\ &= \int \mathrm{d}\boldsymbol{v}' \int \mathrm{d}\boldsymbol{v}\, K(\boldsymbol{v}, \boldsymbol{v}') f(\boldsymbol{v}') \bar{\phi}(\boldsymbol{v}) - \int \mathrm{d}\boldsymbol{v}\, \nu(\boldsymbol{v}) f(\boldsymbol{v}) \bar{\phi}(\boldsymbol{v}) \\ &= \int \mathrm{d}\boldsymbol{v} f(\boldsymbol{v}) \Big(\int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}', \boldsymbol{v}) \bar{\phi}(\boldsymbol{v}') - \nu(\boldsymbol{v}) \bar{\phi}(\boldsymbol{v}) \Big) \\ &= \langle f, J' \phi \rangle, \end{aligned} \tag{2.25}$$

where

$$(J'\phi)(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}', \boldsymbol{v}) \phi(\boldsymbol{v}') - \nu(\boldsymbol{v}) \phi(\boldsymbol{v}). \tag{2.26}$$

Since $\phi(\boldsymbol{v})$ vanishes for v larger than some $\rho > 0$ and $\nu(\boldsymbol{v})$ is continuous it follows that

$\nu\phi \in \mathcal{M}$. Also

$$\left|\int \mathrm{d}\boldsymbol{v}' \, K(\boldsymbol{v}', \boldsymbol{v})\phi(\boldsymbol{v}')\right|$$

$$\leq \|\phi\|_\infty \int_{v' \leq \rho} \mathrm{d}\boldsymbol{v}' \, K(\boldsymbol{v}', \boldsymbol{v})$$

$$\leq n_2 m_2 (m_1 + m_2)^{-1} (\alpha/\pi)^{1/2} \|\bar{\sigma}\|_\infty \|\phi\|_\infty \int_{v' \leq \rho} \mathrm{d}\boldsymbol{v}' \, |\boldsymbol{v} - \boldsymbol{v}'|^{-1} \tag{2.27}$$

where $\|\bar{\sigma}\|_\infty = \sup \bar{\sigma}(\boldsymbol{v}, \boldsymbol{v}')$; $\boldsymbol{v}, \boldsymbol{v}' \in \mathbb{R}^3$. Since the right-hand side tends to zero for large $\boldsymbol{v}$, we conclude that $J'\phi \in \mathcal{M}$ for b-simple ϕ. Thus $J'_{\mathcal{M}}$, the restriction of J' to $\mathcal{M}$ with the b-simple functions as domain, is densely defined in $\mathcal{M}$. Since $J' \subset J^*$ it follows that J^* is defined on a dense set in $\mathcal{M}$. According to (2.23) J^* is also defined when acting upon the constant functions from $\mathcal{X}^*$ and $J^*\phi = 0$ for such ϕ. It is not clear, however, whether or not J^* is in general defined on a dense set in $\mathcal{X}^*$. (This is, of course, the case for bounded J and J^*.) If J^* is densely defined then J is closable (Kato 1966, chap 3, theorem 5.28). In fact J is closable but we have to follow a more roundabout way to arrive at this result. Let $J^*_{\mathcal{M}}$ be the restriction of J^* to $\mathcal{M}$, i.e. its domain is $\mathcal{D}(J^*_{\mathcal{M}}) = \{\phi \in \mathcal{D}(J^*) \cap \mathcal{M} | J^*\phi \in \mathcal{M}\}$. $J^*_{\mathcal{M}}$ is closed along with J^* as is easily checked. $J'_{\mathcal{M}} \subset J^*_{\mathcal{M}}$ so that $J^*_{\mathcal{M}}$ is densely defined in $\mathcal{M}$. It follows that its adjoint $(J^*_{\mathcal{M}})^*$ exists as a closed operator acting in $\mathcal{M}^*$. Let $f \in \mathcal{D}(J) \subset \mathcal{X} \subset \mathcal{M}^*$ and let $\phi \in \mathcal{D}(J'_{\mathcal{M}})$ (i.e. ϕ is b-simple). Then

$$\langle Jf, \phi \rangle = \langle f, J'_{\mathcal{M}}\phi \rangle, \tag{2.28}$$

i.e. J and $J'_{\mathcal{M}}$ are adjoint to each other. Thus J is a restriction of $(J'_{\mathcal{M}})^*$ (Kato 1966, p 167). Since $\mathcal{X}$ is a closed subspace of $\mathcal{M}^*$ it now follows that J has a closed extension $\bar{J}$ acting in $\mathcal{X}$. Conversely, since J is densely defined in $\mathcal{X}$ and J and $J'_{\mathcal{M}}$ are adjoint to each other $J'_{\mathcal{M}}$ is a restriction of J^*. Since $\mathcal{M}$ is a closed subspace of $\mathcal{X}^*$, $J'_{\mathcal{M}}$ has a closure $\bar{J}'_{\mathcal{M}}$ acting in $\mathcal{M}$ and its maximal closed extension is $J^*_{\mathcal{M}}$. Thus we arrive at:

Proposition 2.3. J is a densely defined, closable operator. We denote its closure (i.e. its smallest closed extension) by $\bar{J}$. The domain of J^* at least contains the linear span of the constant function and the b-simple functions. $J'_{\mathcal{M}}$ is closable in $\mathcal{M}$ with maximal closed extension $J^*_{\mathcal{M}}$.

Corollary 2.1. The spectra $\sigma(\bar{J})$ and $\sigma(J^*)$ coincide. Isolated eigenvalues of $\bar{J}$ of finite multiplicity are also eigenvalues of J^* with the same algebraic and geometric multiplicities (and vice versa).

Proof. Since $\bar{J}$ and J^* are real operators (i.e. $\overline{(\bar{J}f)(\boldsymbol{v}_1)} = \bar{J}\overline{f(\boldsymbol{v}_1)}$) their spectra are invariant under complex conjugation (i.e. $\lambda \in \sigma(\bar{J}) \Leftrightarrow \bar{\lambda} \in \sigma(\bar{J})$). The rest follows from (Kato 1966, chap 3, theorem 6.22 and remark 6.23).

Remark. $\mathcal{D}(\bar{J})$ may be larger than $\mathcal{D}(\nu)$ due to cancellation effects between K and ν. This in fact happens with J^*. Since $((1+\nu)^{-1})^*(\boldsymbol{v}) = (1+\nu(\boldsymbol{v}))^{-1}$ is bounded as an operator acting in $\mathcal{X}^*$, ν^* is closed. For $\nu(\boldsymbol{v})$ tending to infinity with v (hard sphere case), $\phi_0 \equiv 1$ is not contained in $\mathcal{D}(\nu^*)$ although $\phi_0 \in \mathcal{D}(J^*)$.

Lemma 2.1. Let $\phi \in L^\infty(\mathbb{R}^3, \mathrm{d}\boldsymbol{v})$. Then $f_1^{(0)}\phi \in \mathcal{D}(J)$ and $\phi \in \mathcal{D}(C)$.

Proof. In all cases that we consider $\nu(\boldsymbol{v})$ grows at most linearly with $\boldsymbol{v}$. Thus $f_1^{(0)}(\boldsymbol{v})\nu(\boldsymbol{v})\phi(\boldsymbol{v})\in\mathscr{X}$ and $\|\nu f_1^{(0)}\phi\|_1\leq\|\phi\|_\infty\|\nu f_1^{(0)}\|_1$. Thus $f_1^{(0)}\phi\in\mathscr{D}(\nu)$ and hence in $\mathscr{D}(J)$. The operator C is defined through

$$(Jf_1^{(0)}\phi)(\boldsymbol{v})=\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v},\boldsymbol{v}')f_1^{(0)}(\boldsymbol{v}')\phi(\boldsymbol{v}')-\nu(\boldsymbol{v})f_1^{(0)}(\boldsymbol{v})\phi(\boldsymbol{v})$$
$$=f_1^{(0)}(\boldsymbol{v})(C\phi)(\boldsymbol{v}). \tag{2.29}$$

Since, as is easily checked,

$$L(\boldsymbol{v},\boldsymbol{v}')\equiv K(\boldsymbol{v},\boldsymbol{v}')f_1^{(0)}(\boldsymbol{v}')=f_1^{(0)}(\boldsymbol{v})K(\boldsymbol{v}',\boldsymbol{v})=L(\boldsymbol{v}',\boldsymbol{v}) \tag{2.30}$$

we have

$$(C\phi)(\boldsymbol{v})=\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})\phi(\boldsymbol{v}')-\nu(\boldsymbol{v})\phi(\boldsymbol{v}). \tag{2.31}$$

Now

$$\|\nu\phi\|_2^2=\int \mathrm{d}\boldsymbol{v}\,f^{(0)}(\boldsymbol{v})\nu(\boldsymbol{v})^2|\phi(\boldsymbol{v})|^2\leq\|\nu\|_2^2\|\phi\|_\infty^2<\infty,$$

whereas for

$$g(\boldsymbol{v})=\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})\phi(\boldsymbol{v}')$$

we have

$$|g(\boldsymbol{v})|\leq\|\phi\|_\infty\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})=\|\phi\|_\infty\nu(\boldsymbol{v}),$$

i.e. $g\in\mathscr{H}$. Thus $C\phi\in\mathscr{H}$ for $\phi\in L^\infty(\mathbb{R}^3,\mathrm{d}\boldsymbol{v})$.

Using the symmetry relation (2.30) for L and the Fubini–Tonelli theorem, we obtain

$$(C\phi,\phi)=\int \mathrm{d}\boldsymbol{v}\,f^{(0)}(\boldsymbol{v})\Big(\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})\phi(\boldsymbol{v})-\nu(\boldsymbol{v})\phi(\boldsymbol{v})\Big)\bar{\phi}(\boldsymbol{v})$$
$$=\int \mathrm{d}\boldsymbol{v}\,f^{(0)}(\boldsymbol{v})\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})[\phi(\boldsymbol{v}')-\phi(\boldsymbol{v})]\bar{\phi}(\boldsymbol{v})$$
$$=\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,L(\boldsymbol{v}',\boldsymbol{v})[\phi(\boldsymbol{v}')-\phi(\boldsymbol{v})]\bar{\phi}(\boldsymbol{v})$$
$$=-\tfrac{1}{2}\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,L(\boldsymbol{v}',\boldsymbol{v})|\phi(\boldsymbol{v}')-\phi(\boldsymbol{v})|^2\leq 0. \tag{2.32}$$

It can be shown that the L^∞ functions are dense in $\mathscr{H}$. Thus (2.32) shows that C is densely defined, non-positive symmetric and consequently has a self-adjoint extension $\bar{C}$.

A comparison of (2.26) and (2.31) shows that J' and C have the same representation. Thus we can hope to obtain some information about the spectrum of J^* from the fact that $\bar{C}$ has real non-positive spectrum. Indeed we have:

Proposition 2.4. The eigenvalues of J^* and the isolated eigenvalues of $\bar{J}$ of finite multiplicity are real and non-positive.

Proof. Let $\phi \in \mathscr{D}(J^*)$ be an eigenfunction of J^* with associated eigenvalue λ. Then

$$\langle Jf_1^{(0)}\phi, \phi\rangle = \langle f_1^{(0)}\phi, J^*\phi\rangle = \bar{\lambda}\langle f_1^{(0)}\phi, \phi\rangle = \bar{\lambda}(\phi, \phi).$$

On the other hand $(\phi \in \mathscr{D}(J^*) \subset \mathscr{X}^* \subset \mathscr{D}(C))$

$$\langle Jf_1^{(0)}\phi, \phi\rangle = \langle f_1^{(0)}C\phi, \phi\rangle = (C\phi, \phi) \leqslant 0,$$

so that λ must be real and non-positive. The remainder follows from corollary 2.1.

3. *J* generates a contraction semi-group

An important question to be answered in the present context is whether or not J (or rather its closure $\bar{J}$) generates a semi-group acting in $\mathscr{X}$. This means that the equation

$$\partial_t f(t) = \bar{J}f(t) \tag{3.1}$$

has a solution in $\mathscr{X}$ once $f \in \mathscr{X}$ is specified at some initial time, say at $t = 0$. Then

$$f(t) = \exp(\bar{J}t)f(0), \tag{3.2}$$

where $\{\exp(\bar{J}t),\ t \geqslant 0\}$ is a semi-group of bounded operators acting in $\mathscr{X}$. Our aim will be to demonstrate that $\{\exp(\bar{J}t),\ t \geqslant 0\}$ is a contraction semi-group of type C_0, i.e. $\exp(\bar{J}t)$ is strongly continuous in $t \geqslant 0$ and

$$\|\exp(\bar{J}t)\|_1 \leqslant 1. \tag{3.3}$$

This is a straightforward matter for $\{\exp(\bar{C}t),\ t \geqslant 0\}$ acting in $\mathscr{H}$ since C is non-positive self-adjoint. One way to ascertain whether or not an operator generates a semi-group is to study its numerical range. In a Hilbert space the numerical range $W(A)$ of an operator A is the set of 'expectation values' (Af, f), $f \in \mathscr{D}(A)$, $\|f\| = 1$. Apart from some other conditions to be imposed upon it, A generates a contraction semi-group if $W(A)$ is contained in the closed left half of the complex plane. This result has been generalised by Lumer (1961) and Lumer and Phillips (1961) to generators of contraction semi-groups in Banach spaces. The starting point is the notion of a semi-inner product (SIP). Then a numerical range can be defined in terms of the former. Thus let f, g, h be contained in a complex Banach space $\mathscr{B}$ with norm $\|\cdot\|$. Then $[f, g]$ is a SIP compatible with the norm in $\mathscr{B}$ if

$$\begin{aligned}
&[f+h, g] = [f, g] + [h, g], \\
&[\lambda f, g] = \lambda[f, g], \qquad \lambda \in \mathbb{C}, \\
&[f, f] = \|f\|^2 \qquad (\text{so that } [f, f] > 0 \Leftrightarrow f \neq 0), \\
&|[f, g]| \leqslant \|f\| \cdot \|g\|.
\end{aligned} \tag{3.4}$$

Any Banach space $\mathscr{B}$ admits at least one SIP compatible with its norm. Its uniqueness depends on the precise topological structure of $\mathscr{B}$.

Let A be an operator acting in $\mathscr{B}$ with domain $\mathscr{D}(A)$. Its numerical range is the set of complex numbers

$$W(A) = \{[Af, f];\ f \in \mathscr{D}(A),\ \|f\| = 1\}. \tag{3.5}$$

A is said to be dissipative with respect to the SIP [,] if $W(A)$ is contained in the closed left half of the complex plane, i.e. the real part of any element of $W(A)$ is non-positive. In general $W(A)$, and hence the dissipativity of A, depends upon the specific SIP under consideration. If A generates a strongly continuous contraction semi-group, however, then A is dissipative with respect to any norm-compatible SIP.

Lemma 3.1. (Lumer and Phillips 1961, lemma 3.4). Let A be densely defined and closable with closure (smallest closed extension) $\bar{A}$. If A is dissipative then there exists a SIP in $\mathscr{B}$ relative to which $\bar{A}$ is dissipative.

Lumer and Phillips (1961, theorem 3.1) have shown that if A is dissipative, densely defined and if the range of $1-A$ is $\mathscr{B}$, then A generates a strongly continuous contraction semi-group. They further state that a closed, densely defined, dissipative operator A generates a strongly continuous contraction semi-group if in addition A^* is dissipative. Since we do not have sufficient control over $\mathscr{D}(J^*)$, we cannot use these results directly. It turns out, however, that $\bar{J}$ and $\bar{J}'_{\mathscr{M}}$ are dissipative and that it is possible to conclude from these results that $\bar{J}$ generates a strongly continuous contraction semi-group.

We introduce a SIP in $\mathscr{X}$. Let $f(\boldsymbol{v})$ and $g(\boldsymbol{v})$ be Lebesgue-measurable complex functions of $\boldsymbol{v}\in\mathbb{R}^3$. We define $\phi_f(\boldsymbol{v})$ by

$$f(\boldsymbol{v})=|f(\boldsymbol{v})|\phi_f(\boldsymbol{v}) \tag{3.6}$$

for $\boldsymbol{v}$ with $f(\boldsymbol{v})\neq 0$ and we set $\phi_f(\boldsymbol{v})=1$ for $\boldsymbol{v}$ with $f(\boldsymbol{v})=0$. $|f(\boldsymbol{v})|$ and $\phi_f(\boldsymbol{v})$ are measurable along with $f(\boldsymbol{v})$ and, since $|\phi_f(\boldsymbol{v})|=1$, $\phi_f(\boldsymbol{v})\in\mathscr{X}^*$. The SIP in $\mathscr{X}$ is now defined by

$$[f,g]_1=\|g\|_1\int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v})\bar{\phi}_g(\boldsymbol{v})=\|g\|_1\cdot\langle f,\phi_g\rangle, \qquad f,g\in\mathscr{X}. \tag{3.7}$$

It is easily checked that the relations (3.4) hold.

Proposition 3.2. $\bar{J}$ is dissipative.

Proof. Let $f\in\mathscr{D}(J)=\mathscr{D}(\nu)$. Then

$$\begin{aligned}
[Jf,f]_1&=\|f\|_1\langle Jf,\phi_f\rangle\\
&=\|f\|_1\int \mathrm{d}\boldsymbol{v}\Big(\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v},\boldsymbol{v}')f(\boldsymbol{v}')-\nu(\boldsymbol{v})f(\boldsymbol{v})\Big)\bar{\phi}_f(\boldsymbol{v})\\
&=\|f\|_1\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,[K(\boldsymbol{v},\boldsymbol{v}')f(\boldsymbol{v}')-K(\boldsymbol{v}',\boldsymbol{v})f(\boldsymbol{v})]\bar{\phi}_f(\boldsymbol{v})\\
&=\|f\|_1\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})f(\boldsymbol{v})[\bar{\phi}_f(\boldsymbol{v}')-\bar{\phi}_f(\boldsymbol{v})]\\
&=\|f\|_1\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\boldsymbol{v}'\,K(\boldsymbol{v}',\boldsymbol{v})|f(\boldsymbol{v})|[\phi_f(\boldsymbol{v})\bar{\phi}_f(\boldsymbol{v}')-1].
\end{aligned} \tag{3.8}$$

Here the Fubini–Tonelli theorem has been applied followed by a relabelling of the variables ($\boldsymbol{v}\leftrightarrow\boldsymbol{v}'$). Since $|\phi_f(\boldsymbol{v})\bar{\phi}_f(\boldsymbol{v}')|=1$ we have $\mathrm{Re}\,\phi_f(\boldsymbol{v})\bar{\phi}_f(\boldsymbol{v}')-1\leqslant 0$. $K(\boldsymbol{v}',\boldsymbol{v})$ being non-negative, it follows that $\mathrm{Re}[Jf,f]_1\leqslant 0$, i.e. J is dissipative. Since $\bar{J}$ is its closure it follows from lemma 3.1 that $\bar{J}$ is dissipative.

Remark. It is seen from (3.8) that $[Jf, f]_1$ vanishes for f with constant phase ϕ_f, for instance for $f(\boldsymbol{v}) \geqslant 0$.

We now turn to the definition of a SIP in $\mathscr{X}^*$. In general it is not easy to give an explicit expression for this quantity. In the following, however, we only need a SIP $[\phi, \psi]_\infty$ for a restricted class of ψ's from $\mathscr{X}^*$. Thus let $\psi(\boldsymbol{v}) \in \mathscr{X}^*$ be such that there exists a $\boldsymbol{v}_0$ for which $|\psi(\boldsymbol{v}_0)| = \|\psi\|_\infty$. This is certainly the case for simple functions ψ and for continuous ψ from $\mathscr{M}$ (the latter tend to zero for large v). We now define ($\phi \in \mathscr{X}^*$):

$$F_\psi(\phi) \equiv [\phi, \psi]_\infty = \phi(\boldsymbol{v}_0)\bar{\psi}(\boldsymbol{v}_0), \tag{3.9}$$

which definition makes sense for every ϕ for which $\phi(\boldsymbol{v}_0)$ is defined, in particular for simple and for continuous ϕ. It is clear that the relations (3.4) hold.

Proposition 3.3. $\bar{J}'_{\mathscr{M}}$ is dissipative.

Proof. Let $\psi \in \mathscr{M}$ be b-simple. We define $[\phi, \psi]_\infty$ according to (3.9). Since $\int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}', \boldsymbol{v})\psi(\boldsymbol{v}')$ is continuous in $\boldsymbol{v}$ for b-simple ψ and since $\nu(\boldsymbol{v})\psi(\boldsymbol{v})$ is defined in $\boldsymbol{v} = \boldsymbol{v}_0$ ($\nu(\boldsymbol{v})$ is continuous), we have

$$\begin{aligned}[J'\psi, \psi]_\infty &= \left(\int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}', \boldsymbol{v}_0)\psi(\boldsymbol{v}') - \nu(\boldsymbol{v}_0)\psi(\boldsymbol{v}_0)\right)\bar{\psi}(\boldsymbol{v}_0)\\ &= \int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}', \boldsymbol{v}_0)[\psi(\boldsymbol{v}') - \psi(\boldsymbol{v}_0)]\bar{\psi}(\boldsymbol{v}_0)\\ &= \int \mathrm{d}\boldsymbol{v}'\, K(\boldsymbol{v}', \boldsymbol{v}_0)[\psi(\boldsymbol{v}')\bar{\psi}(\boldsymbol{v}_0) - \|\psi\|_\infty^2].\end{aligned}$$

Since $K(\boldsymbol{v}', \boldsymbol{v}_0) \geqslant 0$ and $|\psi(\boldsymbol{v}')\bar{\psi}(\boldsymbol{v}_0)| \leqslant \|\psi\|_\infty^2$ it follows that $\mathrm{Re}[J'_{\mathscr{M}}\psi, \psi]_\infty \leqslant 0$. $J'_{\mathscr{M}}$ is a (densely defined) restriction of $J^*_{\mathscr{M}}$ and consequently it follows from lemma 3.1 that $\bar{J}'_{\mathscr{M}}$ is dissipative.

Lemma 3.2. Let $\mathscr{L} \subset \mathscr{X}$ be a non-trivial (i.e. $\mathscr{L} \neq \varnothing$, $\mathscr{L} \neq \mathscr{X}$) closed linear subspace. There exists a $\phi \in \mathscr{D}(J'_{\mathscr{M}})$, $\phi \neq 0$ with $\langle f, \phi\rangle = 0$ for every $f \in \mathscr{L}$.

Proof. Let f be measurable and let P_ρ, $\rho > 0$ be the projector defined by $(P_\rho f)(\boldsymbol{v}) = f(\boldsymbol{v})$, $v \leqslant \rho$ and $(P_\rho f)(\boldsymbol{v}) = 0$ otherwise. Further, let $\mathscr{X}_\rho = L^1(S(\rho), \mathrm{d}\boldsymbol{v})$, where $S(\rho)$ is the ball with radius ρ centred in the origin in $\mathbb{R}^3$. Thus $\mathscr{X}^*_\rho = L^\infty(S(\rho), \mathrm{d}\boldsymbol{v})$. We distinguish two cases.

(a) $\exists \rho > 0$ with $\mathscr{L} \subset \mathscr{X}_\rho$. Let $\phi(\boldsymbol{v}) = 1$ for $2\rho \leqslant v \leqslant 3\rho$ and $\phi(\boldsymbol{v}) = 0$ otherwise. Then $\phi(\boldsymbol{v})$ fulfils the requirements of the lemma.

(b) There is no $\rho > 0$ such that $\mathscr{L} \subset \mathscr{X}_\rho$. Then $P_\rho\mathscr{L}$ is a non-trivial closed subspace of $\mathscr{X}_\rho$. (Since $\|P_\rho f\|_1 \to \|f\|_1$ for $\rho \to \infty$ it follows by taking ρ sufficiently large that we can achieve that $P_\rho\mathscr{L} \neq \varnothing$.) It is a consequence of the Hahn–Banach theorem that there exists a non-zero $\phi \in \mathscr{X}^*_\rho$ with $\langle f, \phi\rangle = 0$, $\forall f \in P_\rho\mathscr{L}$. We extend ϕ to $\mathscr{X}^*$ by setting $\phi(\boldsymbol{v}) = 0$, $v > \rho$. Now ϕ is contained in $\mathscr{D}(J'_{\mathscr{M}})$, $\|\phi\|_{\mathscr{M}} = \|\phi\|_\infty \neq 0$ and for every $f \in \mathscr{L}$ we have $\langle f, \phi\rangle = \langle P_\rho f, P_\rho\phi\rangle = 0$.

Theorem 3.1. $\bar{J}$ generates a strongly continuous contraction semi-group $\{U(t) = \exp(\bar{J}t),\ t \geqslant 0\}$. Thus $f(t) = U(t)f$ is strongly continuous in $t \geqslant 0$ and $\|U(t)\| \leqslant 1$.

Proof. We follow the line of reasoning of the original proof (Lumer and Phillips 1961, § 3). Thus we have to ascertain that $(1-\bar{J})\mathscr{D}(\bar{J})=\mathscr{X}$. Suppose that $\mathscr{L}=(1-\bar{J})\mathscr{D}(\bar{J})\neq\mathscr{X}$. Since $\bar{J}$ is closed dissipative with non-zero domain, $\mathscr{L}$ is closed and non-empty. According to lemma 3.2 there is a $\phi\in\mathscr{D}(J'_{\mathscr{M}})$, $\phi\neq 0$ such that $\langle(1-\bar{J})f,\phi\rangle=0$ for every $f\in\mathscr{D}(\bar{J})$. It follows that $(1-J'_{\mathscr{M}})\phi=0$ but this is impossible since $J'_{\mathscr{M}}$ is dissipative.

Starting from $U(t)$ we can define the adjoint semi-group $U^*(t)$ acting in $\mathscr{X}^*$ by duality. However, although $U^*(t)$ is weak*-continuous, it need not be continuous in the strong topology of $\mathscr{X}^*$. This problem has been considered by Hille and Phillips (1957, ch 14) who introduce a different notion of adjointness, the so-called $\odot$-adjointness. Thus $\mathscr{X}^{\odot}=\overline{\mathscr{D}(J^*)}\subset\mathscr{X}^*$ and the restriction $J^{\odot}$ of J^* to $\mathscr{X}^{\odot}$ generates a strongly continuous contraction semi-group acting in $\mathscr{X}^{\odot}$ which coincides with the restriction of $U^*(t)$ to $\mathscr{X}^{\odot}$. For bounded J, $\mathscr{X}^{\odot}=\mathscr{X}^*$ but for unbounded J this may not be true. Nevertheless, since J^* is defined for the b-simple functions and the constant function, $\mathscr{X}^{\odot}$ at least contains their closed linear span.

Theorem 3.2. Let $f\in\mathscr{X}$ be real, non-negative (i.e. $f(\boldsymbol{v})\geqslant 0$, $\boldsymbol{v}\in\mathbb{R}^3$). Then $f(t)=\exp(\bar{J}t)f$ is also real, non-negative and $\|f(t)\|_1=\|f\|_1$.

Proof. $\phi_0(\boldsymbol{v})\equiv 1$ is contained in $\mathscr{D}(J^{\odot})\subset\mathscr{X}^*$ and is an eigenfunction to $J^{\odot}$ with corresponding eigenvalue zero ($J^{\odot}\phi_0=0$). Consequently $\exp(J^{\odot}t)\phi_0=\phi_0$ so that $\langle f(t),\phi_0\rangle=\langle\exp(\bar{J}t)f,\phi_0\rangle=\langle f,\exp(J^{\odot}t)\phi_0\rangle=\langle f,\phi_0\rangle=\|f\|_1$ since f is non-negative. Let $f^+(\boldsymbol{v},t)=f(\boldsymbol{v},t)$ for $\boldsymbol{v}$ with $f(\boldsymbol{v},t)\geqslant 0$ and $f^+(\boldsymbol{v},t)=0$ otherwise. Further let $-f^-(\boldsymbol{v},t)=f(\boldsymbol{v},t)-f^+(\boldsymbol{v},t)$. (Since $\bar{J}$ is a real operator $f(\boldsymbol{v},t)$ is real along with $f(\boldsymbol{v})$.) Thus $f(\boldsymbol{v},t)=f^+(\boldsymbol{v},t)-f^-(\boldsymbol{v},t)$ is decomposed into its non-negative and negative parts and

$$\langle f(t),\phi_0\rangle=\int \mathrm{d}\boldsymbol{v}\,f^+(\boldsymbol{v},t)-\int \mathrm{d}\boldsymbol{v}\,f^-(\boldsymbol{v},t)=\|f^+(t)\|_1-\|f^-(t)\|_1=\|f\|_1.$$

On the other hand

$$\|f(t)\|_1=\|f^+(t)\|_1+\|f^-(t)\|_1=\|\exp(\bar{J}t)f\|_1\leqslant\|f\|_1.$$

Consequently $\|f^-(t)\|_1=0$, i.e. $f(\boldsymbol{v},t)\geqslant 0$ (almost everywhere, to be precise) and $\|f(t)\|_1=\|f\|_1$.

Remark. The function $f_1(\boldsymbol{v}_1)$ in equation (1.1) is a distribution function, i.e. non-negative. Theorem 3.2 states that for spatially homogeneous systems this property is conserved in time. In addition the normalisation is time-independent.

So far we have not included the case that an acceleration term is present. Let us therefore consider the operator ($\boldsymbol{a}$ is a constant vector)

$$J(\boldsymbol{a})=J-\boldsymbol{a}\cdot\partial_{\boldsymbol{v}_1} \tag{3.10}$$

which replaces J in (3.1). Our procedure will be basically the same as before. There is one difference, however. Since the acceleration term contains the derivative $\partial_{\boldsymbol{v}_1}$ we can no longer make use of a dense set of simple functions. Instead we take a set $\mathscr{N}$ of continuously differentiable functions which is dense in $\mathscr{X}$ and in $\mathscr{M}$. We suppose that $\overline{f(\boldsymbol{v})}\in\mathscr{N}$ along with $f(\boldsymbol{v})$ and that $f(\boldsymbol{v})$ vanishes for sufficiently large $\boldsymbol{v}$. This implies that $\boldsymbol{a}\cdot\partial_{\boldsymbol{v}}f(\boldsymbol{v})$ also vanishes for large $\boldsymbol{v}$. (For $\mathscr{N}$ we may take $C_0^{\infty}(\mathbb{R}^3)$, the space of infinitely differentiable functions of compact support.) Thus $J(\boldsymbol{a})$ with domain $\mathscr{D}(J(\boldsymbol{a}))=$

$\mathscr{D}(J) \cap \mathscr{D}(\boldsymbol{a} \cdot \partial_v)$ is densely defined since each $f \in \mathcal{N}$ is contained in $\mathscr{D}(J(\boldsymbol{a}))$. It follows that $J^*(\boldsymbol{a})$ exists as a closed operator acting in $\mathscr{X}^*$. Since $\phi \in \mathcal{N}$ vanishes for large $\boldsymbol{v}$, $J'\phi$ is contained in $\mathcal{M}$. Also $\boldsymbol{a} \cdot \partial_v \phi$ vanishes for large $\boldsymbol{v}$, i.e. $\boldsymbol{a} \cdot \partial_v \phi \in \mathcal{M}$. Thus

$$(J'(\boldsymbol{a}))_{\mathcal{M}} = J' + \boldsymbol{a} \cdot \partial_v \tag{3.11}$$

with domain $\mathcal{N}$ is densely defined in $\mathcal{M}$. Let $f \in \mathscr{D}(J(\boldsymbol{a}))$ and $\phi \in \mathcal{N}$. Then

$$\langle J(a)f, \phi\rangle = \langle f, (J' + \boldsymbol{a} \cdot \partial_v)\phi\rangle = \langle f, J'(\boldsymbol{a})\phi\rangle = \langle f, (J'(\boldsymbol{a}))_{\mathcal{M}}\phi\rangle \tag{3.12}$$

so that $(J'(\boldsymbol{a}))\mathcal{M}$ is a restriction of $(J^*(\boldsymbol{a}))_{\mathcal{M}}$. In the same way as in § 2 we can now conclude that $J(\boldsymbol{a})$ is closable with closure $\bar{J}(\boldsymbol{a})$. Of course $\bar{J}$ and $\bar{J}(\boldsymbol{a})$, $\boldsymbol{a} \neq 0$, may have different domains. This happens for bounded J; $\mathscr{D}(J) = \mathscr{X}$ but, since $\boldsymbol{a} \cdot \partial_v$ is unbounded, $\mathscr{D}(\bar{J}(\boldsymbol{a})) = \mathscr{D}(\overline{\boldsymbol{a} \cdot \partial_v})$.

We can repeat the line of reasoning followed in the proofs of propositions 3.2 and 3.3, with the result that $[Jf, f]_1$ and $[J'\psi, \psi]_\infty$ have non-positive real parts for $f, \psi \in \mathcal{N}$. Thus it remains to consider $[-\boldsymbol{a} \cdot \partial_v f, f]_1$ and $[\boldsymbol{a} \cdot \partial_v \psi, \psi]_\infty$ for $f, \psi \in \mathcal{N}$. We have

$$\begin{aligned}
[-\boldsymbol{a} \cdot \partial_v f, f]_1 &= \|f\|_1 \int \mathrm{d}\boldsymbol{v}\, \{-\boldsymbol{a} \cdot \partial_v[|f(\boldsymbol{v})|\phi_f(\boldsymbol{v})]\}\bar{\phi}_f(\boldsymbol{v}) \\
&= \|f\|_1 \int \mathrm{d}\boldsymbol{v}\, \{-\boldsymbol{a} \cdot \partial_v |f(\boldsymbol{v})| - |f(\boldsymbol{v})|[\boldsymbol{a} \cdot \partial_v \phi_f(\boldsymbol{v})]\bar{\phi}_f(\boldsymbol{v})\} \\
&= -\|f\|_1 \int \mathrm{d}\boldsymbol{v}\, \overline{f(\boldsymbol{v})}\boldsymbol{a} \cdot \partial_v \phi_f(\boldsymbol{v}) \\
&= \|f\|_1 \int \mathrm{d}\boldsymbol{v}\, \{\boldsymbol{a} \cdot \partial_v \overline{f(\boldsymbol{v})}\}\phi_f(\boldsymbol{v}) \\
&= -\overline{[-\boldsymbol{a} \cdot \partial_v f, f]}_1,
\end{aligned} \tag{3.13}$$

so that (3.13) is purely imaginary as is the case for $[\boldsymbol{a} \cdot \partial_v \phi, \phi]_\infty$ since

$$\begin{aligned}
[\boldsymbol{a} \cdot \partial_v \phi, \phi]_\infty &= [\boldsymbol{a} \cdot \partial_v \phi(\boldsymbol{v})]\bar{\phi}(\boldsymbol{v})|_{\boldsymbol{v}=\boldsymbol{v}_0} \\
&= \boldsymbol{a} \cdot \partial_v |\phi(\boldsymbol{v})|^2|_{\boldsymbol{v}=\boldsymbol{v}_0} - [\boldsymbol{a} \cdot \partial_v \bar{\phi}(\boldsymbol{v})]\phi(\boldsymbol{v})|_{\boldsymbol{v}=\boldsymbol{v}_0} \\
&= -\overline{[\boldsymbol{a} \cdot \partial_v \phi, \phi]}_\infty.
\end{aligned} \tag{3.14}$$

(Note that $\partial_v |\phi(\boldsymbol{v})|^2|_{\boldsymbol{v}=\boldsymbol{v}_0} = 0$ since $|\phi(\boldsymbol{v})|$ takes on its maximum $\|\phi\|_\infty$ in $\boldsymbol{v} = \boldsymbol{v}_0$.) It follows that $J(\boldsymbol{a})$ and $J'_{\mathcal{M}}(\boldsymbol{a})$ both with domain $\mathcal{N}$ are dissipative. Since both operators are closable it follows that their respective closures $\bar{J}(\boldsymbol{a})$ and $\bar{J}'_{\mathcal{M}}(\boldsymbol{a})$ are also dissipative. But now theorem 3.1 can be extended.

Theorem 3.3. Theorem 3.1 holds with $\bar{J}$ replaced by $\bar{J}(\boldsymbol{a})$.

Since $\phi_0(\boldsymbol{v}) \equiv 1$ is still an eigenfunction of $J^*(\boldsymbol{a})$ with corresponding eigenvalue zero, we can repeat the line of reasoning that resulted in theorem 3.2, i.e.

Theorem 3.4. Theorem 3.2 holds with $\bar{J}$ replaced by $\bar{J}(\boldsymbol{a})$.

Remark. The fact that $-\boldsymbol{a} \cdot \partial_{v_1}$ has purely imaginary numerical range is connected with the fact that this operator generates a group of norm-preserving operators in $\mathscr{X}$ and $\mathscr{X}^*$ respectively (Lumer 1964).

Remark. At this point we would have run into serious problems if we had tried to treat the corresponding extension $C(\boldsymbol{a})$ of C acting in $\mathscr{H}$. $C(\boldsymbol{a})$ would have taken the form

$$C(\boldsymbol{a})=C-\boldsymbol{a}\cdot\partial_{v_1}+m\boldsymbol{a}\cdot\boldsymbol{v}_1/(kT)$$

and the operator $-\boldsymbol{a}\cdot\partial_v+m\boldsymbol{a}\cdot\boldsymbol{v}_1/(kT)$ has a spectrum that covers the whole complex plane.

Since zero is an eigenvalue of $J^*(\boldsymbol{a})$ it is also a point of the spectrum of $\bar{J}(\boldsymbol{a})$, but it is not necessarily an eigenvalue ($f_1^{(0)}$ is not an eigenfunction for this eigenvalue; $Jf_1^{(0)}=0$ but $-\boldsymbol{a}\cdot\partial_{v_1}f_1^{(0)}\neq 0$). It can be shown that $\bar{J}(\boldsymbol{a})$ does not have the eigenvalue zero for the fixed scatterer (Lorentz) model. In general the question can be raised whether or not $f(t)=\exp[\bar{J}(\boldsymbol{a})t]f$ tends to a limit for large t. Here we touch upon the question of the long time or ergodic behaviour of $f(t)$ which is in general a complicated matter. We hope to come back to this problem on another occasion.

4. Discussion

In the present work we have shown that the theory of semi-inner product spaces can conveniently be used to demonstrate that for spatially homogeneous systems the Boltzmann equation (1.1) makes sense as an evolution equation acting in the space of absolutely integrable functions of velocity. We also found that the positivity and normalisation of the distribution function are conserved in time.

We did not study the spectral properties of the collision operator J. Our only result in this direction is proposition 2.4 where we stated that discrete eigenvalues of J^* are real. We would expect, however, that the full spectrum of J (and hence of J^*) is real non-positive. This property can indeed be established for a large class of cross sections $\sigma(\boldsymbol{g};\boldsymbol{g}')$ and we intend to discuss these matters in a future publication.

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Appendix

According to equation (2.6),

$$K(\boldsymbol{v}_1,\boldsymbol{v}_1')=\int \mathrm{d}\boldsymbol{v}_2\,\sigma(\boldsymbol{g};\boldsymbol{g}')\delta(\tfrac{1}{2}g^2-\tfrac{1}{2}g'^2)(\alpha/\pi)^{3/2}\exp(-\alpha v_2'^2). \tag{A.1}$$

We set

$$g = w + \Delta, \qquad g' = w - \Delta. \tag{A.2}$$

Momentum conservation (equation (2.6)) yields

$$v_2' = v_2 + (m_1/m_2)(v_1 - v_1') \tag{A.3}$$

so that ($M = m_1 + m_2$)

$$\Delta = [M/(2m_2)](v_1 - v_1'), \qquad w = v_1 - v_2 - \Delta = v_1' - v_2' + \Delta. \tag{A.4}$$

Thus

$$K(v_1, v_1') = \int \mathrm{d}w\, \sigma(w + \Delta, w - \Delta)\delta(2\Delta \cdot w)(\alpha/\pi)^{3/2} \exp[-\alpha(w - v_1' - \Delta)^2. \tag{A.5}$$

Taking Δ parallel to the Z axis we have $v_{11} = v_{11}'$, $v_{12} = v_{12}'$ and $\delta(2\Delta \cdot w) = (2\Delta)^{-1}\delta(w_3)$ and we obtain

$$K(v_1, v_1') = \bar{\sigma}(v_1, v_1')(2\Delta)^{-1}(\alpha/\pi)^{1/2} \exp[-\alpha\Delta^{-2}(\Delta^2 + \Delta \cdot v_1')^2] \tag{A.6}$$

with $\bar{\sigma}(v_1, v_1')$ given by equation (2.8).

Furthermore we find by changing the integration variables from v_1 and v_2 to g and v_2' that

$$\begin{aligned}\int \mathrm{d}v_1\, K(v_1, v_1') &= \int \mathrm{d}v_1\, \mathrm{d}v_2\, \sigma(g, g')\delta(\tfrac{1}{2}g^2 - \tfrac{1}{2}g'^2)f_2^{(0)}(v_2') \\ &= \int \mathrm{d}g\, \mathrm{d}v_2'\, \sigma(g, g')\delta(\tfrac{1}{2}g^2 - \tfrac{1}{2}g'^2)f_2^{(0)}(v_2') \\ &= \int \mathrm{d}v_2'\, g'\sigma_{\mathrm{tot}}(g')f_2^{(0)}(v_2') = \nu(v_1')\end{aligned} \tag{A.7}$$

which is equation (2.15).

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